DOCUMENT RESUME

ED 106 124 SE 019 083

AUTHOR	Cooney, Thomas J.; And Others
TITLE	Protocol Materials in Mathematics Education: Selection of Concepts. Report No. 1.
INSTITUTION	National Center for the Development of Training Materials in Teacher Education, Bloomington, Ind.
SPONS AGENCY	National Center for Improvement of Educational Systems (DHEW/OE), Washington, D. C.
PUB DATE	Feb 75
NOTE	48p.
EDRS PRICE	MF-$0.76 HC-$1.95 PLUS POSTAGE
DESCRIPTORS	*Concept Teaching; Curriculum Development; Educational Theories; *Literature Reviews; *Mathematical Concepts; *Mathematics Education; Teacher Education; *Teaching Methods

ABSTRACT

The basic premise of this report is that in addition to knowledge of subject matter and psychological principles, teachers should have the ability to think about the subject matter and to perform the logical operations used in manipulating it. The basic objects of mathematics instruction are concepts and principles, and, therefore, the focus of this report is the definition of moves and strategies in operating on and teaching these. The "concept of a concept" is discussed, and a taxonomy of concepts defined. Eight types of connotative moves and six types of denotative moves are defined and exemplified. Strategies (sequences of moves) for teaching concepts are then discussed. The nature and importance of principles in mathematics are discussed, and several methods of characterizing principles are compared. Six basic moves for the teaching of principles are identified and associated with key expressions. Rationale for selecting concepts and principles is discussed. (SD)

Protocol Materials in Mathematics Education:
Selection of Concepts

Thomas J. Cooney
Robert Kansky
Kenneth A. Retzer

Report # 7, February, 1975

Foreword[1]

It is possible to identify at least two bases upon which protocol and training materials are currently being developed. One of these can be described as logical or psychological, often based upon an analysis of the teaching or learning process. The second of these is based on an analysis of the subject matter, wherein the objectives in that subject matter are translated into behavioral objectives from which teaching acts are derived. The first of these strategies assumes, questionably, that the teaching-learning process is uniform across all subject matter areas. The second strategy never fully explains the analysis by which objectives are derived in the first place, thus begging a primary question.

As a result of these two quite different orientations, we get two quite different kinds of protocol and training materials. Analysis of the teaching act leads predictably to the development of protocol and training materials that emphasize such concepts as probing, approving, questioning and problem setting. The resulting protocol and training materials very often consist of filmclips from actual classroom settings with concepts and skills developed and tested in simulated or live situations. On the other hand, reducing subject matter to behavioral objectives predictably leads to the development of protocol and training materials emphasizing such concepts or skills as assessment, mastery, sequencing of instruction, planning instruction, and measuring outcomes. The materials are much more likely to be in the form of paper and pencil activities with evaluation either in "live" settings or by means of printed instruments.

The paper which follows is different. Like the second strategy above, it seems to be based on an analysis of a particular subject matter area, in this case mathematics. The position argues that the kinds of concepts and principles in mathematics are probably peculiar to mathematics. Thus, the analysis is epistemological rather than behavioral. It also argues that the particular characteristics of the concepts and principles in mathematics may be correlated with teaching moves designed with those particular concepts in mind. The authors' reasoning seems to be this: one begins with development of a taxonomy of the particular kinds of concepts that seem to have importance in the mathematics curriculum. For example, there are denotative concepts and attributive concepts. The former are either concrete or abstract and, within each of those categories, either singular or general. Given this classification system, which the authors define very specifically, it is possible to identify and classify many different mathematical concepts. The teaching of those concepts becomes an objective. The notion of a "teaching move" is introduced and particular teaching moves are derived, depending upon the kind of concept. A "strategy" then is a sequencing of teaching moves. For mathematical principles, as opposed to concepts, a simplified taxonomy is created, "moves" are again derived from categories of mathematical principles, and finally "strategies" are developed from "moves."

---

[1]The original study that resulted in this paper was supported by U.S. Office of Education funds through the Leadership Training Institute in Educational Personnel Development at the University of South Florida, Tampa. The authors of this paper are (in the order their names appear) at the University of Georgia, University of Wyoming and Illinois State University.

It would seem to be obvious that the protocol materials developed from this kind of orientation would not lead either to materials on such teaching behaviors as approving, probing or questioning, or to materials focused on such objectives as assessment, mastery or planning. Rather, the result would be materials useful in identifying attributive concepts, concrete concepts, etc., within the subject matter and identifying such teaching moves as a "single characteristic move" or "connotative move." In terms of skills, one would expect students to become skilled in planning particular moves and accurately making the moves in live situations. It would also suggest that the important higher-order skills would involve implementing teaching strategies consisting of moves based upon an analysis of the particular subject matter that is being taught at the time.

The authors make no claim regarding the generality of their model. It is up to developers of training and protocol materials in other subject matter areas to review this model and to imagine its implications if applied to other areas of science, social studies, etc. It is an imaginative, creative and refreshing departure from the strategies that presently seem to be dominating efforts to develop protocol and training materials in teacher education. The editors commend it to your attention.

L.D. Brown, Editor

Table of Contents

Page

Protocol Materials in Mathematics Education:
Selection of Concepts

Thomas J. Cooney
Robert Kansky
Kenneth A. Retzer

The two major purposes of this paper are (1.) to identify concepts which are important in the teaching of mathematics, including those whose application to the teaching of mathematics is significantly different from their application in other subject matter areas and (2.) to suggest certain of those concepts for possible protocol materials development.

Our search for concepts in mathematics education will focus upon the teaching of mathematics. It will, furthermore, be based upon an epistemological theory of teaching since such organized bodies of knowledge can provide a basis for both conceptualizing and exemplifying various aspects of teaching mathematics.

Survey of the Literature

A Theory of Teaching Mathematics Based on Epistemology

It is generally recognized that extensive knowledge of both subject matter and basic psychological principles applicable to classroom instruction are essential aspects of a teacher's knowledge base. What is not as readily recognized is the fact that there is another type of knowledge that is basic to effective instruction. As Smith (1969) puts it:

> It has only recently been recognized that there is another sort of knowledge that can influence the performance of the teacher: that used in thinking about the subject matter and the logical operations used in manipulating it (p. 125).

One way to conceive of logical operations for manipulating mathematics is to focus on objects of instruction, e.g., mathematical concepts and principles. From this perspective one can conceptualize various teaching acts, i.e., moves, for manipulating these types of mathematical knowledge.

One should realize that the objects of instruction are themselves a determining factor in the way one manipulates the content being taught.

Smith (1969) writes:

> The subject matter of each field of teaching is a mixture of different forms of knowledge. All of the fields contain concepts. Some contain laws or law-like statements. Others contain rules and theorems and still others include values, either as major emphasis or as incidental to other forms of content. It is important for the teacher to be aware of these knowledge forms because studies have shown that each is taught and learned in a different way (p. 127).

Smith further states:

> Faculties in the various disciplines have failed to analyze the content of instruction into its logically and pedagogically significant elements. Teachers who are ignorant of these elements do not know how to handle their content other than by common sense. A teacher cannot handle with skill a form of knowledge he cannot identify and whose structure he does not know (p. 127).

Thus, it seems reasonable to assume that knowledge relevant to the nature of mathematics and of moves used in manipulating the objects of instruction are highly relevant to teachers of mathematics.

This report will focus on pedagogical concepts for teaching mathematical concepts and principles. An explication of mathematical concepts and principles and of moves for teaching these types of knowledge will be given. To begin, let us consider the nature of a mathematical concept.

A Concept of *Concept*

Van Engen (1953) pointed to the confusion concerning the meaning of the term *concept*. Cooney, Davis and Henderson (in press) note that one possible explanation for this confusion is that "concepts are complex objects, and attempts to identify all of their salient characteristics have so far been unsuccessful." Since the meaning of *concept* is central to this report, it will be necessary to examine some points of view regarding a concept of *concept*.

Many educators, educational psychologists and philosophers define *concept* from the standpoint of concept formation; thus a concept is an abstraction. It is the "characteristics or properties common to a set of objects which in their concrete form are different in many respects (Wehlage and Anderson, 1972, p. 17)." It is a kind of screen for separating examples from nonexamples. While an identifying term may be assigned to a concept, a term is unnecessary to the existence of a concept (abstraction).

Psychologists, such as Bruner (1956), point out that a concept evokes common verbal or nonverbal responses which a person makes to similar (not necessarily identical) stimuli. Hence, to Bruner concept formation is a categorization process; a concept is, therefore, a category:

> To categorize is to render descriminably different things equivalent, to group the objects and events around us into classes, and to respond to them in terms of their class membership rather than their uniqueness (p. 1).

While naming is only one form of response (verbal), it is important to Bruner that this response form be consistent when it is used. The term employed in such a response is arbitrary; the important condition is that the same term be used in response to similar stimuli. In addition, a person's nonverbal responses may be in such forms as use (e.g., blowing of a whistle) or rejection (e.g., avoidance of a hot object). These kinds of responses are also acceptable evidence that a person has a given concept.

Still other theorists regard a concept as the meaning of the term used to designate the concept, that is, a statement of the conditions under which the term may be employed. Thus, teaching a concept is equivalent to teaching how to use a term which designates the concept.

Wehlage and Anderson (1972) identify two basic definitions of *concept*. It is either a mentalistic container which is comprised of "those characteristics thought to be properties common and jointly peculiar to the denotation of a particular set of objects, event, or the like" or "all of the associations one has with a term (p. 18)." In the latter case, a concept may be a personal construction whose characteristics include not only those associated by definition but also those associated by individual value systems or experience. For instance, the set of characteristics associated with *horse* may include not only "four-legged mammal" but also "something of value."

The two definitions of concept given by Wehlage and Anderson (1972) are based upon two quite different rationales. The writers note that the "view that concepts are all of the associations one has with a term implies an emphasis on the personally unique (p. 31)." By contrast, the view that a concept is a set of common properties suggests a focus on similarities and "directs one to focus on limitation and logical implication (p. 31)."

According to Henderson (1967; 1970), a concept is an ordered pair. One component of that ordered pair is a "designatory expression" and the other is a set of rules for using the designatory expression (1970). He considers the teaching of a concept to a student to be the same as teaching the student how to use the designatory expression (1967), and subsequently identifies three uses of designatory expressions (1970):

> *Connotative*. The term is used to talk about the properties its referents have in common or the conditions under which the term would be properly applied.
>
> *Denotative*. The term is used to identify examples and nonexamples of the concept, that is, of members or nonmembers of the referent set of the concept.
>
> *Implicative*. Metalanguage is used to talk about the term itself (rather than its referents) (p. 171).

The first two uses take place in the object language; the third requires metalanguage in order that the term may be assigned a meaning through either a stipulated-definition move or a reported-definition move (Henderson, 1970).

Henderson's concept of *concept* has lent itself to both an analysis of the concepts of mathematics and also the identification and organization of concepts in the teaching of mathematical concepts. Because of the extensive analysis in the latter area, Henderson's point of view of *concepts* would seem to have the greatest potential of providing a basis for the development of protocol materials.

Mathematical Concepts

Unlike Brownell (1941) and his followers, who base their analysis of teaching on a broadly defined meaningful-drill dichotomy associated with the "how" of teaching mathematics, Henderson (1969) begins with a partitioning of the mathematical knowledge to be taught into concepts and principles. Henderson (1970) presents a taxonomy of concepts based upon how a concept is used. His first subdivision of concepts is into three classes:

> *Denotative*. A concept which has a nonempty referent set (i.e., for which there is at least one example). For instance, the numbers 2 and 4 are in the referent set of the denotative concept *even number*.

> *Nondenotative*. A concept which has an empty referent set (i.e., for which there is no example). For example, *perfect number less than six* has an empty referent set.
>
> *Attributive*. A concept which is not a set selector. The concepts *rigor, truth* and *justice* are attributive. While one acknowledges the existence of these concepts, it is not possible to point to an object or event and say "That is rigor," or "That is truth," or "That is justice (p. 173)."

The teaching of denotative concepts accounts for a large part of instruction in mathematics; it is accomplished, generally speaking, by giving examples. Attributive concepts are not taught directly, but rather by inference from correlated denotative concepts; meaning is inferred from use. Nondenotative concepts ordinarily are not taught. They arise from "man's linguistic ability to put words together in meaningful expressions"(Cooney et al., in press). In view of the preceding statements, it is understandable that nondenotative concepts play no significant role in Henderson's taxonomy of concepts or in the analysis of teaching which follows. Moreover, denotative concepts receive more attention than attributive concepts since the former are taught directly, whereas the latter are taught indirectly through the use of their denotative correlates.

For denotative concepts, Henderson (1970) identifies two second-order subclasses:

> *Concrete*. A concept for which elements of the referent set have such physical attributes as size, shape, color, mass, volume or location (in space or time).
>
> *Abstract*. A concept for which elements of the referent set do not have observable properties such as size, shape, color, mass, volume or location (in space or time) (pp. 174-75).

For both denotative and attributive concepts, Henderson (1970) identifies two other subclasses:

> *Singular*. A concept having exactly one element in the referent set.
>
> *General*. A concept having more than one element in the referent set (pp. 175-76).

These are second-order subclasses of attributive concepts and third-order subclasses of denotative concepts. The tree diagram of the taxonomy given in Figure 1 is that of Henderson (1970).

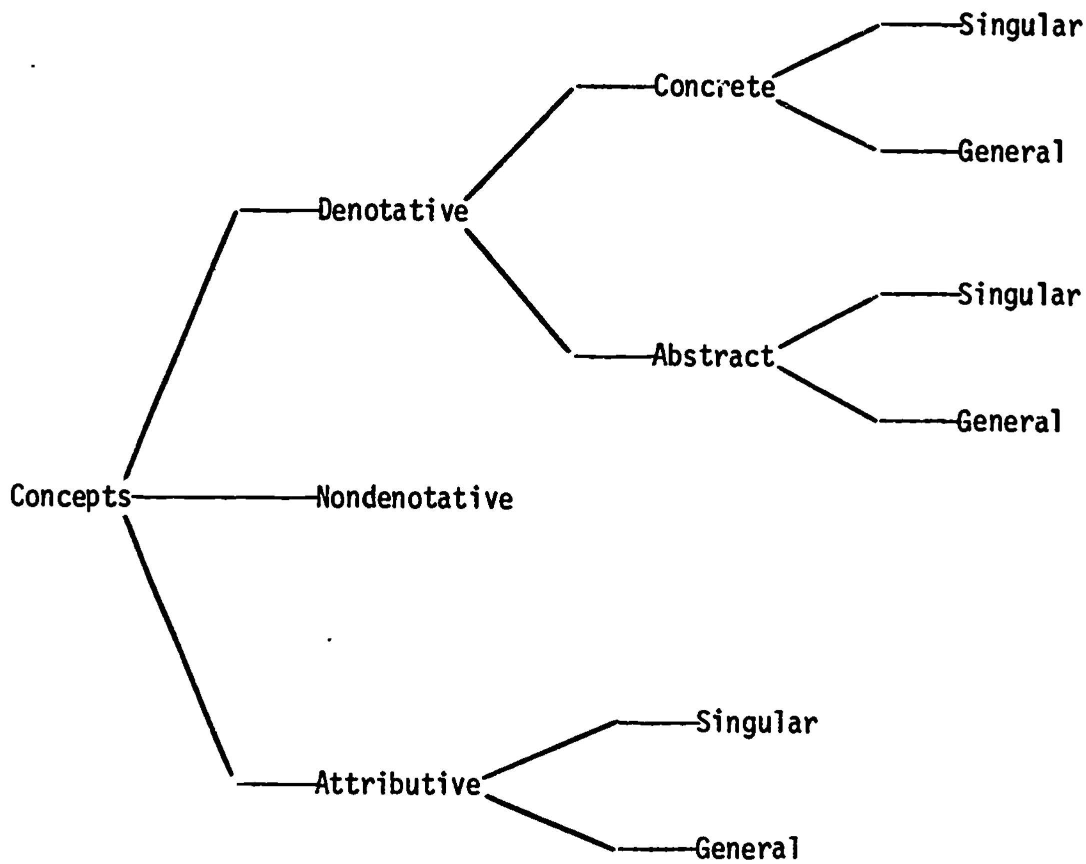


Figure 1. Taxonomy of Concepts

This taxonomy could be appended to include additional subclasses such as simple/complex or precise/vague. However, the practical value of such appending is unknown. Henderson (1970) observes:

> How far the classification is extended by any person probably depends on whether the extension is functional, that is, whether the extension can be correlated with possible or actual differences in how teachers teach the various concepts (p. 176).

Since extension is, then, a practical matter in constructing a theory of teaching and since Henderson's study of classroom episodes does not lead him to suggest the need to add to the seven categories in the taxonomy of Figure 1, it seems ill-advised to complicate the taxonomy until such time as use of the teaching model gives evidence that further subdivisions might be useful.

The Importance of Concepts

The importance of concepts and, hence, the teaching of concepts has been attested to by various scholars. Some argue that concepts are the foundation for all other cognitive knowledge. Generalizations and rules of procedure are based upon them. Bruner (1956) specifies a variety of reasons why concepts are important objects of instruction. The following reasons are usually cited by Bruner and others as indications of why concepts are important:

a. reduce complexity by classifying;

b. aid in identifying (and, hence, discriminating among) objects in the environment;

c. aid in communication;

d. reduce the amount to be learned by making available class labels;

e. permit the drawing of inferences about a labeled object;

f. provide a basis for further learning by aiding in forming generalizations;

g. assist with reasoning and explaining, and;

h. help in organizing data.

It follows that conveying knowledge to students about concepts is essential in the teaching of mathematics. Therefore, some of the means of teaching mathematical concepts are promising subjects for protocols. Mathematics teachers should be able both to observe and effect actions which teach these concepts.

Teaching Mathematical Concepts

To enable one to discuss ways in which mathematical *concepts* can be taught, Henderson (1970) employs the notion of a move, a move being "a sequence of verbal behaviors by a teacher and students toward attaining some objective (p. 177)." The construct of *moves* as conceived by Smith et al.(1964), was used in the analysis of tapes of classroom teaching in grades 8 through 12. The names given to various moves by Henderson are somewhat different from those given by Smith; Henderson's names will be used here simply because they have been selected with an eye to the teaching of mathematics.

Henderson's first division of concept moves distinguishes moves in the object language from moves in the metalanguage. The former deal with the teaching of denotative or attributive *concepts;* the latter are definition moves which deal with the implicative use of a term. The outline of moves used in teaching a concept which is to be presented here follows that given by Henderson (1970). A tree diagram of this outline is given in Figure 2 and serves as a guide to, and summary of, the detailed outline which appears below.

In preparing that detailed outline, the authors borrowed definitions and examples given by Henderson (1970) and by Cooney et al. (in press). Following the name and definition of each move, sentences are listed which exemplify that move. For simplicity those sentences are not listed in quotation marks even though each example is a sample verbalization of a text, teacher or student.

I. Moves in the object language

   A. *Connotative moves.* A connotative move is one which is based on the connotation of the term designating the concept. It focuses upon the conditions connoted by the term.

      When applied to a denotative concept, a connotative move is one in which a textbook, teacher or student talks about the characteristics or properties of the objects in the referent set determined by the concept.

      When applied to an attributive concept, a connotative move is one in which a person states or talks about the conditions that are relevant for the term to be applied

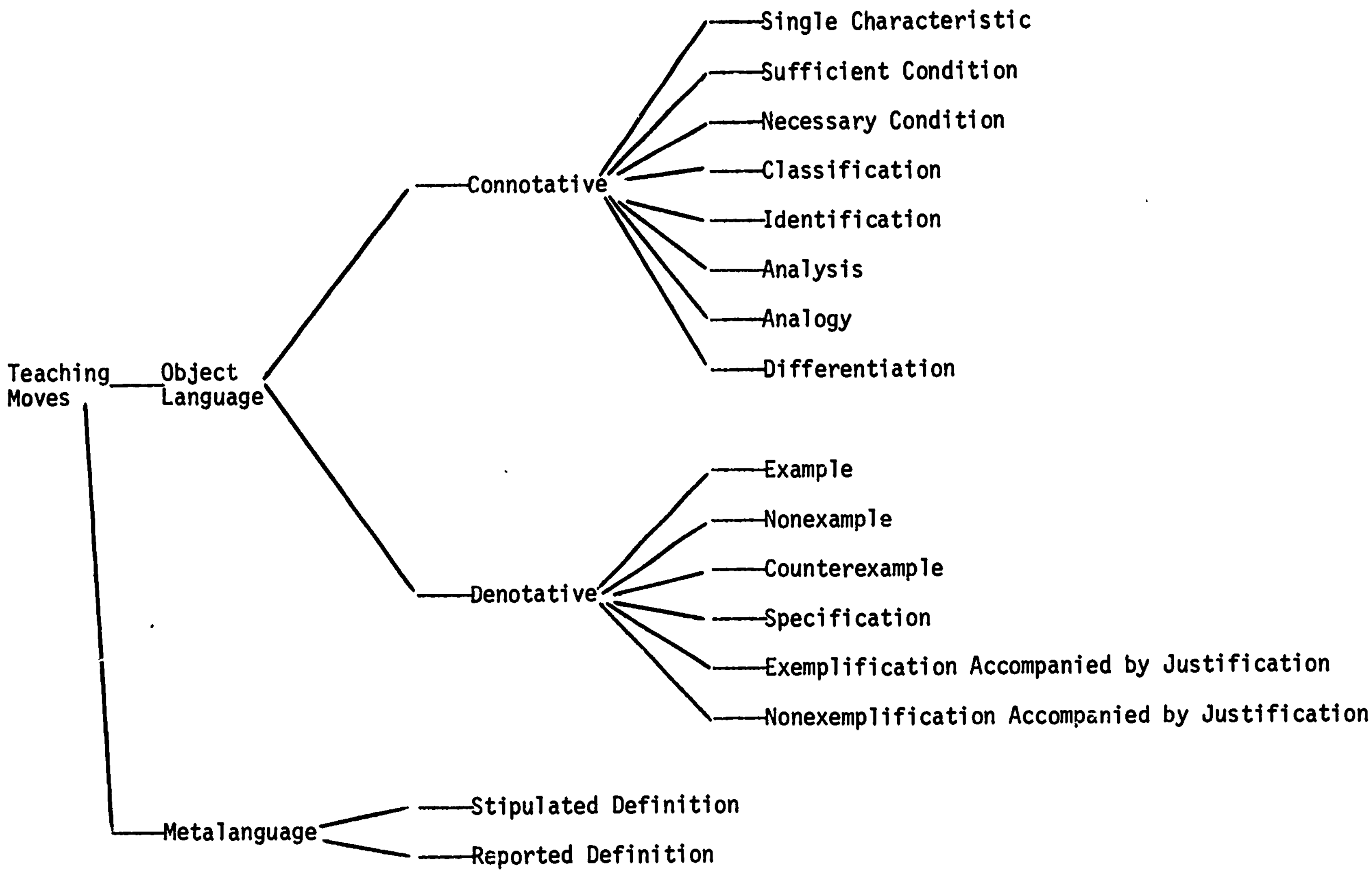


Figure 2. Taxonomy of Concept Teaching Moves

to something.

1. *Single characteristic move*. When applied to a denotative concept, a single characteristic move is one which states one characteristic or property of the element (s) of the referent set.

   A *rhombus* is equilateral.
   Trigonometric functions are periodic.

   When applied to an attributive concept, a single characteristic move is one which states one relevant condition for applying the designating term.

   To be called *prime*, a number must be greater than 1.

2. *Sufficient condition move*. When applied to a denotative concept, a sufficient condition move states, or discusses, a condition (or conjunction of conditions) sufficient to determine an object's membership in the referent set of the term.

   If a parallelogram is equilateral then it is a *rhombus*.
   This rhombus is a *square* because it is equiangular.
   Knowing that a numeral ends in 2 or 4 is sufficient to tell one that the number named is an *even number*.
   If each of two angles is a right angle, the angles are *congruent angles*.

   When applied to an attributive concept, a sufficient condition move states or discusses a condition (or conjunction of conditions) sufficient for applying the term.

   If a number has only itself and 1 as positive, integral divisors, then it can be called *prime*.

3. *Necessary condition move*. When applied to a denotative concept, a necessary condition move states or discusses a property which is necessary for an object to be in the referent set of the term.

   A quadrilateral is a *rhombus* only if it is equilateral.

If a figure is a *trapezoid,* the sum of the measures of its interior angles must be 360 degrees.

When applied to an attributive concept, a necessary condition move states a condition necessary for applying the term.

For a number to be called *square* it must have only even exponents in its prime factorization.
If a name for a number ends in 7, the number cannot be called *even*.

4. *Classification move*. A classification move is one in which a superset of the referent set of the term is identified, discussed or elicited by a question. The word *is,* when used in a classification move, is the *is* of set inclusion or set membership and is followed by a predicate noun.

   A *trapezoid* is a quadrilateral.
   Area is a denominate number.
   Every *prism* is a polyhedron.

5. *Identification move*. (Necessary and Sufficient Condition Move). An identification move states the conditions which are necessary and sufficient for an object to be in the referent set of a term. The word *is,* when used in an identification move, is the *is* of identity.

   A *square* is an equiangular rhombus
   A number having exactly two factors is a *prime number*.
   A relation f is a *function* if and only if no two ordered pairs in f have the same first component.

6. *Analysis move*. An analysis move names, describes or elicits by a question one or more subsets of the referent set of the term. It is not necessary that the collection of subsets identified constitute a partition of the referent set of the term.

   The circle, parabola, ellipse and hyperbola are *conic sections*.

Kites and rectangles are *quadrilaterals*.

7. *Analogy move*. When applied to a denotative concept, an analogy move is one which states that a referent of the term is like something else or discusses in what way the referent is like something else.

   The *system of integers modulo 7* is like the rational number system since each is a field.
   The equations 'X + 2 = X' and 'X - 2 = X' are alike in that each has an empty solution set.
   A *square* is like a triangle in that each is a rectilinear figure.

   When applied to an attributive concept, an analogy move states that one property is like another or discusses in what way the properties are alike.

   Associativity is like commutativity in that each deals with a single binary operation.

8. *Differentiation move*. When applied to a denotative concept, a differentiation move is one in which a referent of the term is said to be different from something else or which discusses the way in which the referent is different.

   The *system of integers* is unlike the system of integers modulo 8 in that the system of integers has no divisors of zero.
   A *trapezoid* differs from a parallelogram in that a trapezoid has exactly one pair of parallel sides.

   When applied to an attributive concept, a differentiation move is one which states that a property is said to be different from another property or which discusses the way in which the properties are different.

   Distributivity is unlike associativity in that distributivity involves two operation..

B. *Denotative moves.* A denotative move is one in which either a teacher, student or textbook names members or nonmembers of the referent set of the concept. Since, by definition, an attributive concept is not a set selector, an attributive concept has no referent set. Hence, denotative moves apply only to denotative concepts.

1. *Example move.* An example move is one in which one or more examples of the referent set are given (stated, drawn, pointed to, acted out, etc). Syntactically, an example move makes use of a proper noun.

   Six is a *perfect number.*
   The numbers 1, 8, 27, 64 and 125 are *cubed numbers.*

2. *Nonexample move.* A nonexample move is one which gives one or more examples of objects which are not in the referent set.

   Twelve is not a *perfect number.*
   $3X^2$ is not a *binomial.*

3. *Counterexample move.* A counterexample move is a move in which an object is designated which falsifies a generalization which purports to characterize the members of the concept's referent set. It is a move which is used in response to the asserting of a false generalization.

   Assertion: Every odd number is prime.
   Response: Name the factors of 9.

   Assertion: Every real number has a reciprocal.
   Response: What's the reciprocal of zero?

4. *Specification move.* A specification move is one which designates each of the members of the referent set.

   The *perfect numbers less than 30* are 6 and 28.
   The *four conic sections* are the circle, the ellipse, the hyperbola and the parabola.

5. *Exemplification accompanied by justification move.* In this move, an example of the referent set is accompanied by a reason why it is an example.

Six is a *perfect number* because its factors (other than 6) are 1, 2 and 3 and the sum of those factors is 6.

6. *Non exemplification accompanied by justification move.* In this move, an example of an object not in the referent set is accompanied by a reason why it is not in the referent set.

   Nine is not a *prime number* because it has three factors (1, 3 and 9) whereas a *prime number* has exactly two factors.

II. Moves in the metalanguage

A. *Stipulated-definition moves.* A stipulated definition move is one in which a teacher encourages his students to invent a name for an idea.

Teacher: Lori has suggested that it would be nice to have a simple name for this relation we've been using which maps each real number onto itself. What shall we call it?

Student: The *do-nothing function.*

Teacher: Good enough. By '*do-nothing function*' we will mean...

B. *Reported-definition moves.* A reported definition move states or elicits the statement of the lexical definition of a term.

The symbol "*iff*" is an abbreviation for "*if and only if.*"
A triangle having exactly one symmetry is called "*an isosceles triangle.*"
A parallelogram is called a "*rhombus*" if and only if the parallelogram is equilateral.

Strategies in Teaching Concepts

A temporal sequence of moves is called a *strategy* (Henderson, 1970). For example, a sequence of definition/exemplification/nonexemplification moves could be used to illustrate an expository teaching strategy. A sequence of example/nonexample/sufficient-condition moves could be used to illustrate a guided discovery teaching strategy.

By identifying the conditions under which a strategy or its component moves may be effective in helping certain kinds of students to learn a specified concept, hypotheses about the efficacy of such teaching strategies can be generated. In fact, there are a few studies which have investigated certain strategies based upon the moves of this model.

Nuthall (1966), in a study using programmed materials with senior high school students, concluded that strategies involving exemplification moves in teaching selected social studies concepts (cultural symbiosis and ethnocentrism) are most effective and that those using comparison moves[1] are least effective. Although one might conjecture that this result would hold for mathematical topics or for instructional modes other than programmed textbooks, these conjectures have not been investigated.

Rollins (1966), also using programmed materials, examined the efficacy of three strategies using examples and nonexamples to teach verbal concepts. One strategy was comprised entirely of example moves, a second strategy used example/nonexample pairs and the third strategy was comprised of a set of example moves followed by a set of nonexample moves. No one of these strategies was found to be more effective than any other across the three ability levels (high, average, low) of eighth grade students in the study.

A third study using programmed materials was done by Rector (1966). Four programmed booklets were designed to teach eleven elementary probability concepts to college freshmen; each booklet employed one of four strategies: characterization, characterization/exemplification, exemplification/characterization, and exemplification/characterization/exemplification. Testing at three levels of awareness (defined behaviorally and using the categories of Bloom [1956]), showed that the strategy consisting entirely of characterization moves (i.e., moves which disclose characteristics of elements of the referent set) was more effective than any of the other three strategies with regard to promoting awareness at the level of knowledge and comprehension. (The knowledge and comprehension level was considered to be the lowest level of understanding of the three defined.)

---

[1] Comparison and/or contrast moves are a subcategory of Henderson's 1967 model for teaching mathematical concepts; they appear as analogy and differentiation moves in the model reported in this paper.

At present, there is no convincing evidence with regard to the special effectiveness of any particular strategy in teaching specified concepts to any homogenous group of persons. The studies are few. Although the three reported here all used a programmed presentation, no two used the same strategies or comparable subjects. As a group, then, these studies suggest little more than that "a teacher is largely free to choose among the various possible strategies in teaching a concept in mathematics (Henderson, 1970, p. 195)." Since no one strategy has been identified by research as being most effective in any given situation, Henderson (1969) concludes that "other things being equal, the teacher may want to use different strategies if for no other reason than to provide variety in his teaching (p. 13)."

A Concept of *Principle*

In reading the literature on teaching, one finds that the term *principle* is frequently used but only rarely defined. The term is used in a variety of ways and its use often results in ambiguity. To avoid this, an explication of this concept will be given. We propose the following definition:

> A *principle* is a true generalization or else an efficacious prescription.

This definition is consistent with Henderson (1969), who notes that *principle* usually refers to a generalization other than an existential generalization or a prescription. To further explicate this concept, we will consider the nature of mathematical generalizations and prescriptions.

The nature of mathematical generalizations. Wills (1970), defines a generalization as a "sentence stating that each element of a specific set has a particular property, or that some elements of the set do (p. 288)." Wills discusses two types of generalizations, universal and existential. To Wills a universal generalization is a "simple sentence that begins with a universal quantifier, symbolized by $\forall$; this means 'for any' and is the beginning of a phrase that precedes a true statement (p. 289)." Similarly, Wills considers an existential generalization as a "simple sentence that begins with the existential quantifier, symbolized by $\exists$; this means 'there is---for which,' and it is followed by the appropriate statement (p. 289)."

Wehlage and Anderson (1972) identify the following characteristics of generalizations:

1. All generalizations are statements asserting a claim.
2. The claim is a relationship between examples of a concept and some specified factor.
3. Sometimes these statements of relationship are asserted in conditional form (If---then).
4. All these statements of relationships imply or directly state a quantification claim.
5. Quantifiers are either uniform or statistical (e.g., all, usually).
6. The nature of the claim asserted can be reversible or irreversible, necessary or substitutable, or sufficient or contingent (p. 58).

Although the authors are discussing generalizations in the context of social studies curriculum, their characteristics are also appropriate for describing most mathematical generalizations.

Exner and Rosskopf (1959) consider a generalization as a combination of a quantifier and a predicate, a predicate being a sentence which indicates a property of elements in a universal set. To Exner and Rosskopf, a universal quantifier is an expression written next to a predicate whose presence indicates that the property symbolized by the predicate holds for all individuals in the universe. Thus, a sentence is a universal generalization when a universal quantifier is used next to a predicate as is illustrated below.

For every triangle, the sum of the degree measures of the interior angles of the triangle is 180.

For every real number x, $\sin^2 x + \cos^2 x = 1$.

Similarly, the authors introduce an *existential quantifier* as an expression which conveys that there is at least one true instance of the predicate. The combination of an existential quantifier and a predicate is an existential generalization.

The following statement exemplifies an existential generalization.

There is at least one natural number n such that the sum of the proper divisors of n is n.

Retzer (1967) maintains that a generalization is recognizable by its form. He stipulates that a universal generalization is a general statement with two component parts, viz., a quantifier and an open sentence. Thus, to Retzer, while the sentence:

All even numbers are divisible by 2

is a general statement, the statement below constitutes a universal generalization:

For each even number x, x is divisible by 2.

The quantifier in the latter statement consists of the phrase "For each even number x;" the clause "x is divisible by 2" is the open sentence.

Retzer further points out that the quantifier contains two bits of information, viz., what the universal set (i.e., the set of objects under consideration) is and which elements of the universal set will produce true instances of the open sentence. In the case of a universal quantifier, the message is that each element of the universal set will produce a true instance; in the case of an existential quantifier, the message is that there is at least one true instance possible.

In school mathematics, the quantifier is often implicit. Examples of such generalizations are given below.

If two sides of a triangle are congruent, then the angles opposite these sides are congruent.

In a proportion, the product of the means is equal to the product of the extremes.

In both cases, the quantifiers are implicit. In the first, the statement holds for all triangles having two congruent sides. Similarly, the second statement pertains to all proportions even though the quantification is not explicit. It might be noted that in both of the given generalizations, a common noun with the indefinite article *a* was used to make the generalization. This is frequently the case when the quantifier is implicit in the general statement.

The generalizations displayed above hold over a single universal set. However, the quantification of some generalizations involves more than one universal set. Consider the generalization given below.

For each convex polygon p and for each natural number $n \geq 3$, if p has n sides then the sum of the degree measures of the interior angles of p is $(n - 2) \times 180$.

This statement holds over two sets, viz., the set of convex polygons and the set of natural numbers greater than 3.

The nature of mathematical prescriptions. Another kind of cognitive knowledge is that of prescriptions. A *prescription* is an order, directive or command. Prescriptions are not statements because they do not have the property of being true or false (Henderson, 1961). Consider the following mathematical prescription.

> To produce a fraction equivalent to a given fraction, multiply the numerator and denominator of the given fraction by any nonzero real number.

This sentence is not declarative; it does not state what is the case. Instead, it is imperative; it directs one to perform a particular operation to attain a certain end.

This suggests a difference in the use, syntax, and, hence, the logic of generalizations and prescriptions. While prescriptions advise or give directions, generalizations are used to declare, report, describe or conjecture what is the case about a set of objects. Furthermore, generalizations are used to make assertions and thus, are truth functional; that is, either the truth-value true or the truth-value false can be assigned to generalizations.

Ordinarily, prescriptions are not considered to have a truth-value. For example, it seems odd to say that the directive:

> To multiply two powers of the same base, add the exponents and use the sum as the exponent of the base of the product

is true. But we can define a truth-function, viz., if following the prescription attains the identified end, assign the truth-value true; if not, assign the truth-value false. Under this rule the prescription above is true.

Now consider the prescription:

> Get all terms containing the unknown by themselves on one side of the equation.

In contrast with the former prescription, this prescription has no explicit identification of the end to be attained. How is a truth-function to be defined? The answer is that one cannot be defined without making some assumptions. But certain assumptions seem reasonable. The prescription is used in a context, most likely that of solving a polynomial equation. Both the teacher and students are aware of the objective, viz., solving the equation; hence, this objective is implicit. Were it to be made explicit, the prescription would become more definite. If we further assume that other operations will be performed on the equation, we can define a truth-function in the same way we define the function for the former prescription.

Consider also the following bits of advice a mathematics teacher might give.

a. To divide two fractions, invert the divisor and multiply.

b. To solve an equation of the form:

$$ax + b = c$$

first subtract b from each member of the equation and then divide each member by a.

c. To solve the equation:

$$2x + 4 = 7$$

first subtract 4 from each member of the equation.

d. To evaluate 2^6, first find 2^3 and then square the result.

e. To evaluate an expression of the form a^{2n}, first evaluate a^n and then square the result.

All of those locutions are prescriptions and truth-values could be assigned as discussed above. Prescriptions a, b and e could be appropriately used by a teacher when instructing students on how to divide fractions, solve equations of a certain form, or raise a given real number to an even integral power, respectively. The directives given in c and d may be regarded as instances of the more general prescriptions given in b and e. While teachers often give specific directives, such as c and d, the directives are based on prescriptions which apply to a more extensive domain. Similarly, when a teacher is showing students how to divide fractions, he may give advice pertaining to a specific example.

We shall call those prescriptions which apply to extensive domains *efficacious prescriptions*. *Efficacious prescriptions* and *true generalizations* constitute what we have termed *principles*. Heretoafter, *prescriptions* and *generalizations* will be used synonymously for *efficacious prescriptions* and *true generalizations*, respectively.

The Importance of Principles

Principles constitute a major component of knowledge taught in secondary school subjects. Wehlage and Anderson (1972) in discussing the importance of generalizations identify four basic functions of generalizations:

1. They serve as conclusions to particular efforts at inquiry.
2. Generalizations also provide the basis for generating new hypotheses, i.e., what else might be true given the implications of a generalization?
3. Generalizations are indispensable in the operations of explanations and prediction.
4. General propositions are basic elements in building scientific theory in that theories consist of interrelated sets of generalizations about generalizations (p. 56).

Generalizations also play a major role in the structure and content of mathematics. Exner and Rosskopf (1959) point out that the basic elements of a mathematical system are as follows:

a. an underlying language;
b. a deductive logic system;
c. a vocabulary of undefined words;
d. a set of axioms, and;
e. theorems.

Two of these components, axioms (statements assumed to be true) and theorems (statements whose verification is based on proof), can be stated as generalizations. Retzer (1967) states:

> The structure of any branch of mathematics contains the axioms which are assumed to be true and the theorems which are proved from them. These axioms and theorems may be stated as generalizations. Thus, generalizations occupy an important central position in the structure of mathematics (p. 1).

In school mathematics, axioms and theorems usually consist of universal generalizations as opposed to other types of generalizations.

Part of the significance of the classification of principles into generalizations and prescriptions comes when one considers the use of these two kinds of principles. Consider the principles:

$$\log x^y = y \log x$$
$$\text{antilog}(\log x) = x$$

whose variables are considered restricted to the proper domains. One might say that anyone who knows these principles can apply them to obtain an approximation of a real number raised to a rational power. But teachers know that many students cannot apply them readily. Hence, teachers teach a principle which is prescriptive in form:

> To calculate the power of a number, multiply the logarithm of the number by the exponent of the power and find the antilogarithm of the product.

Students can apply the prescriptive form more readily. In general, the implications for action of principles which are prescriptions are clearer than those of principles which are generalizations. As Henderson (1969) puts it:

> The implication for behavior of a prescription or algorithm is explicit; this is the pedagogical advantage of a prescription. In contrast, a slow learner may not know what to do after he has been taught the correlative generalization. Hence, prescriptions are useful in teaching skills (p. 14).

Since prescriptive principles advise and direct, they are of primary importance in teaching skills. Although logically a student can become skilled in performing a task solely by imitating the actions of another, it is not a pedagogically sound practice for the acquisition of skills to be based solely on imitation. Prescriptions can provide direction for students who might be confused; they can make algorithms explicit for students and provide a basis for understanding the procedure involved in a given algorithm. In short, they can be a facilitating factor in students' acquiring mathematical skills. While skills are not usually considered cognitive knowledge, they are a basic component of what is taught and learned in school mathematics. The acquisition and maintenance of skills is essential for students involved in studying mathematics.

Because of the importance of learning mathematical principles, it seems highly relevant to consider moves a teacher can use in teaching this type of knowledge. We turn now to the consideration of teaching mathematical principles.

Teaching Mathematical Principles

A pedagogical model for teaching mathematical principles has been proposed by Henderson (1969). For the most part the model was conceived through interactions of observing classroom behavior and logical considerations. We shall utilize this model in describing moves used in teaching principles.

Henderson (1969) identifies four basic aspects of teaching principles:

a. stating the principle;
b. clarifying the principle;
c. justifying the principle, and;
d. applying the principle.

In the first category, Henderson identifies three means by which a principle can be brought to the attention of students. First, the principle may be stated outright. Secondly, the principle may be referred to in the textbook. Finally, it may be educed from students who have been engaged in a guided discovery lesson.

There are, according to Henderson (1969), five ways in which a principle can be clarified. These are:

a. paraphrasing the principle;
b. reviewing the meaning of terms in the principle;
c. analyzing the principle into its components, i.e., identifying its hypothesis and conclusion;
d. giving instances, provided the principle is a generalization, and;
e. demonstrating application, provided the principle is a prescription (pp. 14-15).

Paraphrasing a principle involves stating the principle in different, assumedly simpler words for students. It might also involve students stating the principle in their own words, perhaps with the aid of a diagram. The second means of clarifying a principle involves hypothesizing or determining through questions which terms in a generalization may be unfamiliar to students and then reviewing the meaning of those terms. For example, suppose students were considering the following generalization:

> The centroid of a triangle divides each median of the triangle into a ratio of 2:1.

It is unlikely that this statement would have meaning for students if they did not know what a centroid or a median is or what it means to divide a line segment into a ratio of 2:1.

The third means of clarifying a principle (in this case a generalization) is to point out the hypothesis of the generalization (when the generalization is expressed in the form of a conditional) and its conclusion. It is important for students to understand these two components if they are to prove the generalization and apply it correctly. Some students are unable to justify a generalization because they cannot decipher what conditions are given and what must be deduced. The explicit identification of the components of an implication stemming from a generalization is an important pedagogical consideration in clarifying generalizations.

Sometimes students misuse a generalization because they are confused as to what conditions must be satisfied before the generalization can be appropriately applied. Consider the Pythagorean Theorem stated below:

> In a right triangle the sum of the squares of the measures of the legs equals the square of the measure of the hypotenuse.

Implicit in this generalization is the implication that "If you have a right triangle, then...." Thus, when a student concludes that a 3-4-5 triangle is a right triangle because $3^2 + 4^2 = 5^2$ he has misused the given generalization. In short, he cannot justifiably infer that such a triangle is a right triangle because the inference scheme implied by the generalization designates only that <u>if</u> the triangle is a right triangle then $3^2 + 4^2 = 5^2$, <u>not</u> conversely.

A fourth method of clarifying principles is the giving of instances in the case where principles are generalizations. Instances of a generalization are obtained by replacing each of the mathematical variables specified in the generalization with a constant. The resulting statements (instances) are truth functional and are necessarily true provided the generalization is true and the constants are selected from the domain specified or implied by the generalization. To illustrate, consider the generalization below (usually referred to as *the communitative law for addition*).

For all real numbers x and y, $x + y = y + x$

Instances of this generalization can be obtained by replacing the variables *x* and *y* with constants, say *3* and *.14*. Thus, the resulting instance would be:

$$3 + .14 = .14 + 3.$$

Similarly, a prescriptive principle can be demonstrated by selecting appropriate constants and using those constants to show how the task indicated in the prescription is to be performed.

The third major category given by Henderson (1969) involves justifying the principle. Wolfe (1969) defines *justification* as follows:

> A binary relation which has as its domain the union of a set of assertions and a set of actions, either performed or contemplated, and which has as its range the union of a set of sets of reasons and a set of sets of entities of evidence (p. 14).

In other words, Wolfe holds that there are two components to the justification of statements. The first is the assertion to be justified and the second is the sets of reasons offered in support of the given assertion.

Wolfe identified four kinds of content that are justified in mathematics classrooms: universal generalizations, existential generalizations, singular statements and proposed or completed actions. He then explicated six kinds of evidence teachers use to support the four types of content:

Subsuming generalization. A universal generalization is quoted of which the assertion is either an instance or a consequence.

Supporting instance. A true instance of a generalization is noted, making the generalization seem more plausible.

Search for a counterinstance. A counterexample which produces a counterinstance of a universal generalization is noted or discussed, or the absence of a counterexample is noted or discussed.

Deductive proof. A deductive proof is presented or discussed.

Pragmatic reason. A pragmatic reason in defense of a proposed or contemplated action is noted or discussed.

Justified algorithm. An algorithm expressing verbally a performed or proposed action or course of action is justified mathematically.

Henderson discusses various ways in which principles can be justified. For principles that are generalizations, a teacher might exhibit, or ask students to exhibit, instances which the students recognize as being true. In conjunction with this exercise, a teacher might challenge students to produce a counterinstance to the generalization. If the students are unable to produce one, then this may be taken as evidence that the generalization is true. For example, if presented the generalization:

> Any even number greater than 4 can be expressed as the sum of two odd primes

students may search in vain for an even number greater than 4 which cannot be expressed as the sum of two odd prime numbers. Hence, the generalization seems believable to them. Of course, this method of justification is not conclusive. That is, one always runs the risk that the next case considered will produce a counterinstance and, hence, disprove the generalization. It should be noted that generating instances for students to consider may in some situations clarify a generalization, while in others it may provide a basis for justification. Perhaps an instance that helps clarify the meaning of a generalization for one student provides evidence for another student to believe the generalization is true.

A more conclusive method of justification is that based on proof, i.e., a deductive argument. Smith and Henderson (1959) identify the following strategies of proof used to justify mathematical statements:

> *Counterexample.* Finding a counterexample is a stratagem which disproves a statement.
>
> *Detaching an antecedent (modus ponens).* This kind of argument fits the inference scheme $[(p \rightarrow q) \wedge p] \rightarrow q$.
>
> *Developing a chain of propositions.* Sometimes we discover that a proposition p can be established if we can prove q and q can be asserted if r is true. For some reason, we know r is true. Hence, by using the following inference scheme we can prove p $[r \wedge (r \rightarrow q) \wedge (q \rightarrow p)] \rightarrow p$. This stratagem is a repeated application of detaching an antecedent stratagem.
>
> *Proving a conditional.* This stratagem involves assuming the antecedent (or its equivalent) and arguing the consequent (or its equivalent).

Reductio ad absurdam. This stratagem consists in accepting the contradictory of the proposition to be proved and proving that this leads to an inconsistent proposition, e.g., $q \wedge \sim q$, which is necessarily false. It then follows that the proposition under consideration is true.

Indirect proof. This stratagem requires the formulation of a true disjunction whose disjuncts are exhaustive of the domain under consideration. Next, each disjunct is proved false except for the given proposition. This kind of argument can be characterized by the following inference scheme:

$$[(p \vee q \vee --- \vee t) \wedge ((\sim q) \wedge --- \wedge (\sim t))] \rightarrow p.$$

Proving a statement of equivalence. This involves proving a proposition equivalent to the proposition to be proven. Assumedly the equivalent proposition is easier to establish.

Mathematical induction. In this stratagem a proposition A consists of an infinite sequence of propositions A_1, A_2, A_3.... To prove A, one proves A_1 and the conditional $A_r \rightarrow A_{r+1}$, where r is some positive integer. Then using the inference scheme indicated in 2 above, the consequent A is established.

The first of these stratagems can be used to disprove a universal generalization. Each of the others are means by which a generalization can be verified in a deductive manner.

Finally, Henderson (1969) points out two ways in which prescriptive principles can be justified. The first involves demonstrating that following the prescription will lead one to the correct answer. This method is analogous to what Wolfe (1969) calls *pragmatic reason.* Showing students how to determine if a number is divisible by 3 or 9 by summing the digits and deciding if this sum is divisible by 3 or 9 is a procedure which is seldom justified to students in terms of a deductive argument. Rather, students become convinced the prescriptive principle is true by virtue of it "working" in every case that it is applied.

The second method relies on showing that the prescription is justified in terms of previously established generalizations. Wolfe refers to this method as *justified algorithm.* Consider the often used prescription, "To divide two fractions, invert the divisor and multiply." This prescription can be shown to follow from basic axioms and definitions of arithmetic, for example:

$\frac{a}{b} \div \frac{d}{c} = \frac{a}{b} \times \frac{1}{\frac{d}{c}}$ Definition of division (i.e., $a \div b$ is defined to be $a \times \frac{1}{b}$, $b \neq 0$).

where $b \neq 0$, $c \neq 0$, and $d \neq 0$

$\frac{a}{b} \times \frac{1}{\frac{d}{c}} = \frac{\frac{a}{b}}{\frac{d}{c}}$ Definition of multiplication of rational numbers.

$\frac{\frac{a}{b}}{\frac{d}{c}} = \frac{\frac{a}{b} \times \frac{c}{d}}{\frac{d}{c} \times \frac{c}{d}}$ Theorem: $\forall a, \forall b \neq 0, \forall c \neq 0$

$$\frac{a}{b} = \frac{ac}{bc}$$

$\frac{\frac{a}{b} \times \frac{c}{d}}{\frac{d}{c} \times \frac{c}{d}} = \frac{\frac{a}{b} \times \frac{c}{d}}{1}$ The product of a number (excluding zero) and its reciprocal is 1.

$\frac{\frac{a}{b} \times \frac{c}{d}}{1} = \frac{a}{b} \times \frac{c}{d}$ Division property of one.

$\therefore \frac{a}{b} \div \frac{d}{c} = \frac{a}{b} \times \frac{c}{d}$ Transitive property of equality.

The last category of moves for teaching principles discussed by Henderson is that of *application*. This involves using the generalization to work exercises, solve problems or to generate additional knowledge.

Cooney et al.(in press) also categorize moves for teaching a generalizations into four basic categories:

1. Introduction
2. Interpretation (clarification)
3. Justification
4. Application.

These categories are quite similar to those explicated earlier by Henderson (1969). Briefly, introducing a generalization involves focusing a student's attention on the generalization, establishing an educational objective involving the generalization and providing motivation for learning.

The second category includes paraphrasing, reviewing any constituent concepts conjectured to be the source of a lack of comprehension, providing instances and analyzing the generalization into its component parts (hypothesis and conclusion).

Justification moves involve the giving of instances, challenging students to produce counterinstances, or providing a deductive argument establishing the generalization.

Applying the generalization is essentially the same as the moves discussed earlier involving application.

Strategies in Teaching Principles

In the preceding section, various moves or logical operations were discussed that are appropriate for teaching principles. From these, a teacher can form sequences of moves and thereby create a strategy for teaching a given principle. For example, a viable strategy might consist of the following moves: focus, motivation, paraphrase, instantiation, justification (proof) and application. Another possible strategy might be the sequence of focus, objective, review, justification (perhaps presenting instances) and application moves. The selection of a viable teaching strategy will depend on the maturity and ability of students, the complexity of the principle and other factors relevant to formulating teaching strategies. At present, there is not adequate research indicating which strategies are most effective.

One can conceptualize a guided discovery approach and an expository approach using the moves explicated above. Typically, a discovery lesson will involve instantiation moves prior to asserting or stating the principle. In this kind of strategy the asserting of the principle is likely to occur near the completion of the lesson, if it appears at all. On the other hand, an expository approach is usually characterized by stating the principle early in the lesson, with other introduction, interpretation, justification and application moves to follow. Usually discovery lessons focus on principles that are generalizations rather than prescriptions.

Listing of Concepts Selected

The selection of concepts for protocol production entailed certain constraints. Concepts sought were to be those of mathematics education rather than mathematics. The concepts were to be, as nearly as possible, unique to the field of mathematics education. Concepts sought were to be identifiable and portrayable in the sense that they can be adequately described, and possess behavioral indicators. Finally, concepts sought were to be in the public domain of literature in mathematics education rather than being of limited usage or obscure.

Clearly, the teaching moves selected are concepts in mathematics education. These concepts are unique to mathematics education only in the following sense. Mathematics has been subjected to an analysis of the types of knowledge which comprise it, viz., concepts, principles, facts and skills. Also, transcripts of classrooms in which those kinds of knowledge were taught have been studied to identify the verbal moves which are appropriately used to teach that type of subject matter. In no other content field have both been done simultaneously.

The concepts suggested for protocol development are a subset of the verbal moves outlined in the first part of this paper and, as such, are clearly definable and in the public domain. These concepts are listed below, along with the indicators which give evidence that they are portrayable. Further discussion on the rationale for selecting these concepts appears in the last section of this paper.

Concept Moves Selected

Six pedagogical concepts (concept moves) are suggested for protocol development with respect to teaching mathematical concepts. Each of the concept moves was explicated previously. Each move in the following list is accompanied by a collection of key words and phrases (or expressions) which would seem to indicate the presence of the associated move:

1. Identification move (Necessary-and-Sufficient Condition Move).

   Key expressions: necessary and sufficient

   if and only if

2. Sufficient Condition Move.

   Key expressions: sufficient
   if-then
   since
   because
   provided that

3. Necessary Condition Move.

   Key expressions: necessary
   has to
   implies
   must
   only if

4. Analogy Move.

   Key expressions: like
   comparable to
   similar to
   same as
   compare

5. Differentiation Move.

   Key expressions: unlike
   differs from
   contrast
   not the same as

6. Counterexample Move.

   Key expressions: consider
   how about
   suppose that

Principle Moves Selected

This section will focus on the various moves that are particularly relevant and important for the teaching of mathematical principles. Since the moves have already been discussed in a previous section, we will focus here on the context in which the moves are likely to occur and/or be exemplified through protocol materials.

Review concepts (interpretation move). Sometimes the use of this move is triggered by a student asking for clarification of the meaning of a particular term in a principle. Or a teacher may, *a priori*, ascertain that students will have difficulty with a particular term in a given principle and, hence, review it.

Analysis: pointing out the hypothesis and conclusion implicit in a generalization (interpretation move). A teacher might use this move in anticipation of student difficulty and/or to simply provide a basis for better understanding a principle that is a generalization. Students, by the nature of their errors, indicate their confusion in understanding the inherent implication of a generalization. In such cases, a teacher might offset the difficulty by using an analysis move. Consider the situations described below:

> Suppose a student is trying to prove the following generalization.
>
> > The quadrilateral formed by joining the midpoints of the adjacent sides of a rectangle is a rhombus.

Often students experience difficulty in proving this and other similar statements because of their inability to establish what conditions are assumed true and what conditions are to be established. In short, they are unable to identify the hypothesis and conclusion of the implied conditional statement. To alleviate or prevent such problems, a teacher can express the generalization as the following conditional statement:

> If a quadrilateral is formed by joining the midpoints of the adjacent sides of a rectangle, then the resulting quadrilateral will be a rhombus.

and, consequently, help students identify the hypothesis and conclusion.

Consider another situation in which a student is given an obtuse triangle with the measures of the shorter sides being 5 and 12. The student proceeds to find the measure of the third side by taking the square root of $5^2 + 12^2$. The student may feel that he is justified in such an approach because of the Pythagorean Theorem. However, such is not the case. What the student has failed to realize is that he must first have a right triangle before he can appropriately apply this theorem. That is, he failed to realize the condition indicated in the hypothesis of the generalization. An example was given earlier of a student misusing a generalization (the Pythagorean Theorem) because he mistakenly used the converse of the theorem.

One might consider an analogous situation in teaching principles that are prescriptions. That is, students can be alerted as to when it is appropriate to utilize a prescription. For example, the prescription:

> Invert the divisor and multiply

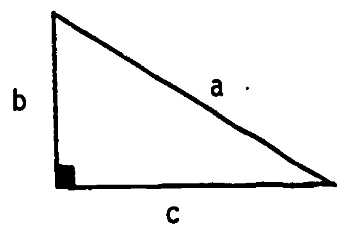
is applied when students are dividing fractions, not multiplying them.

Similarly, the prescription:

To square a number multiply it by itself

is followed when one wishes to square a number, not double the number or find its square root. By identifying the conditions in which a prescription can be appropriately applied, the analysis move has utility when applied to prescriptive principles.

Paraphrasing (interpretation move). This move could occur any time a teacher felt it useful to state or have a student state the principle in terms of less complex language, or in terms of a diagram. For example, it might be insightful to have a student state the Pythagorean Theorem in terms of the diagram below:



Should a student state the usual equation "$a^2 + b^2 = c^2$", then it is unlikely he understands the theorem. On the other hand, should the student, in paraphrasing the theorem, state the correct equation "$b^2 + c^2 = a^2$", then one would be more justified in concluding the student understands the theorem.

Prescriptive principles, too, can be paraphrased. Students can explain in their own words how to follow a given algorithm indicated by one or more prescriptions.

Instantiation (interpretation or justification move). As discussed previously, an instance of a generalization is obtained by replacing each of the variables in the generalization with appropriate constants. The logical aspect of producing instances is the same regardless of their purpose. Whether an instance is for clarification, or for justification purposes, cannot always be determined.

However, there are some verbal cues which may signal the reason an instantiation move occurs. For example, if a student is grappling with the generalization:

One less than the square of any odd number is divisible by 8 and claims he does not understand it, then he is probably asking for an instance to help him interpret the generalization. If a teacher generates an instance using the number 5 ($5^2 - 1 = 24$ and 24 is divisible by 8), the student may be satisfied. On the other hand, if a student questions whether or not the generalization holds for all odd numbers, then he is probably seeking justification. If such is the case, then examining instances involving a wide sampling of odd numbers (e.g., positive, negative, small, large) may convince him the generalization is true. If not, then a proof may be required.

When the concept of instantiation is applied to prescriptions the result is essentially the selection of an element from the proper domain and the demonstration of the prescription. Given the usual prescription for dividing two fractions (stated above), an instantiation move would involve selecting two fractions and demonstrating the prescription utilizing the selected fractions.

Verbal cues indicating the need for instantiation moves for interpretation include:

> Let us consider the case where...
> Let us take a specific case.
> For example,...

Wolfe (1969) identifies several verbal cues which indicate that justification is desired. Among these are:

> Show me the statement is true.
> How do you know that?
> Can you prove it?
> Convince me it's true.
> Why did you do it that way?

These statements seem to suggest that justification is required, either by instantiation or by some other means.

Searching for a counterexample (justification move). A teacher might use this method if students were unable to grasp a deductive proof. Or students might contend that a certain generalization is false, whereupon a teacher might respond:

> Well, if it is false, then you should be able to give me a counterexample. Can you?

If students are unable to produce one, the generalization is likely to become more believable to them.

Proof: deductive and pragmatic (for prescriptive principles). Various strategies for a deductive proof were discussed earlier. Generally, formal mathematical arguments are reserved for students more sophisticated in their ability to cope with inference schemes. However, the context in which proofs are given is essentially the same as that for producing instances. The basic element is the desire to show that a principle is true.

For a prescriptive principle, a deductive proof involves establishing that the prescription in question is a logical consequence of established generalizations. A pragmatic "proof" is showing students that the end result of following a prescription is correct. For example, students might ask "Why do we bisect a line segment that way?" Such a question seems to indicate that justification for a given procedure is being sought. A teacher might respond to this question by showing that the procedure is based on previously learned material, or by measuring the two constructed segments and determining if they appear to be of equal measure. In some cases, a teacher may wish to include both methods of justification for a prescription.

An alternative to providing a complete deductive argument is to indicate to students how a proof might be done, but not present the argument *in toto*. This is effective when students are mature enough to perceive how the argument would proceed once its outline is presented.

Applying the principle (application move). Applying a principle, or indicating to students how a principle might be used, can occur in a number of ways. The most common context occurs when exercises are provided in which the principle is utilized. For example, students who have learned the principle:

For all real numbers a, b, and c such that $a \neq 0$,
if $ax^2 + bx + c = 0$, then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

might then be required to apply it in solving equations such as:

$$2x^2 - 4x + 3 = 0.$$

They might also be asked to utilize the principle to produce another principle:

The sum of the roots of a quadratic equation
written as $ax^2 + bx + c = 0$ is $\frac{-b}{a}$

In short, principles can be applied in solving problems or in generating knowledge previously unknown to students.

Rationale for Selection of the Concepts

It has been previously argued that concepts and principles are of primary importance in mathematics. It seems clear, then, that the learning of mathematical concepts and principles is essential if mathematics is to be meaningful for students. Because of this, it seems highly relevant to identify and explicate teacher actions dealing with the teaching of mathematical concepts and principles and to convey these pedagogical notions to mathematics teachers. This type of knowledge should be a part of the knowledge base which teachers can draw upon. In reference to this type of knowledge, Smith (1969) makes the following observations:

> The teacher who possesses this sort of knowledge and language has an extra dimension from which to observe his own teaching behavior as well as that of his pupils... Because teachers do not now possess such understanding, they frequently handle the subject matter of instruction in superficial ways. Consequently, class discussion often suffers from undue vagueness and ambiguity, from unfounded and unchallenged claims, from a failure to develop the significance of the content (p. 126).

These considerations might lead one to suggest that each move in the taxonomies for teaching mathematics concepts and principles should have been suggested for protocol development. Each of them makes a significant contribution to the teaching of mathematics and, as indicated earlier, is unique to mathematics education in the sense that the matching of pedagogical moves with the content to be taught has reached a stage of maturity only in mathematics education literature.

Nevertheless, it seems plausible that some of these moves would be equally applicable to other content fields. For example, it is difficult to visualize a teacher presenting a social studies concept who could not give examples of that concept and, hence, make use of exemplification moves. Similarly, a science teacher introducing a principle of physics might very plausibly tell the class how this principle leads to other principles, and, hence, make use of a motivation move. Therefore, the claim is not made that these moves are unique to mathematics education even though they surfaced during work that was done uniquely in this field. A lesser claim is made, that is, that taxonomies of moves for teaching mathematics concepts or principles contain concepts which are important to mathematics education. It can be left to subsequent research to determine their relative importance as these moves are embedded in strategies which are shown to have a maximum effect on learning.

Even though these pedagogical moves are not unique to the teaching of mathematics, those selected for initial protocol development appear to be less useful in other subject matter fields because mathematics is a body of analytic knowledge. Contrasted with a body of contingent knowledge whose truths are established by observations in the external world, mathematical truths depend on logical equivalences and sequences of valid inferences in which precision of language plays an important role. Consequently, the concepts selected either aid in the careful exposition important to the teaching of mathematics or else aid in approaching these concepts and truths informally. Thus, we can make the following comments with respect to the concepts selected for initial protocol development.

Necessary and/or sufficient conditions for mathematical concepts can be precisely identified in a way in which they cannot be for concepts in other academic areas. Because mathematical concepts are precise, characteristics that are both common and unique to various concepts can be discussed. Deductive methods of justifying principles have no counterpart in other academic disciplines common to the public schools. The application of principles in establishing additional mathematical knowledge is a consequence of mathematics being a precisely defined body of knowledge. The identifying of the hypothesis and the conclusion of a generalization stated as a conditional statement is not unique to teaching mathematics, but it does take on particular significance when proving and applying mathematical generalizations. The generation of instances, necessarily true providing certain conditions are satisfied, is a characteristic unique to mathematics. Thus, many of the moves identified above are unique and/or particularly important in teaching mathematical generalizations.

Thus, the moves for teaching mathematical concepts and principles identified in the preceding section are justifiable as pedagogical concepts to be illustrated by protocol materials. They ought to be a part of the knowledge mathematics teachers have in planning lessons, in teaching and in evaluating students' progress.

References

Bloom, B. S., (Ed.) Taxonomy of educational objectives, the classification of educational goals. Handbook 1. Cognitive domain. New York: David McKay Co., 1956.

Brownell, W. A. Psychological considerations in the learning and teaching of arithmetic. In Tenth yearbook of the National Council of Teachers of Mathematics. Washington, D.C.: National Council of Teachers of Mathematics, 1941.

Brownell, W. A. When is arithmetic meaningful? Educational Research, 1945, 38, 481-498.

Brownell, W. A. Arithmetical abstractions--Progress toward maturity of concepts under differing programs of instruction. Arithmetic Teacher, 1963, 10, 322-328.

Bruner, J., Goodnow, J., & Austin, G. A study of thinking. New York: John Wiley & Sons, 1956.

Commission on mathematics for the College Entrance Examination Board. Program for college preparatory mathematics. New York: College Entrance Board, 1959.

Committee on the undergraduate program in mathematics, Mathematical Association of America, Recommendations for the training of teachers of mathematics. Washington, D.C.: Mathematical Association of America, 1966.

Cooney, T., Davis, E., & Henderson, K. Dynamics of teaching secondary school mathematics. Boston: Houghton, Mifflin, in press.

Diliberto, S. P. Some remarks on mathematics and the teaching of mathematics. In D. Friedman (Ed.), Proceedings of the UICSM conference on the role of applications in a secondary school mathematics curriculum. Urbana, Ill.: University of Illinois Committee on School Mathematics, 1964.

Exner, R. M. & Rosskopf, M. Logic in elementary mathematics. New York: McGraw-Hill, 1959.

Goals for school mathematics: The report of the Cambridge Conference on School Mathematics. Boston: Houghton, Mifflin, 1963.

Henderson, K. B. Abstracting, generalizing, and explaining--Processes or relations? Mathematics Teacher, 1961, 54, 600-605.

Henderson, K. B. A model for teaching mathematical concepts. Mathematics Teacher, 1967, 60, 573-577.

Henderson, K. B. Teaching secondary school mathematics. Washington, D.C.: National Education Association, 1969.

Henderson, K. B. Concepts. In Thirty-third yearbook of the National Council of Teachers of Mathematics. Washington, D.C.: National Council of Teachers of Mathematics, 1970.

Hendrix, G. Learning by discovery. Mathematics Teacher, 1961, 54, 290-299.

Koenker, R. H. Psychology applied to the teaching of arithmetic. Arithmetic Teacher, 1958, 5, 261-264.

Lankford, F. G., Jr. & Pattishall, E. G., Jr. Experimental study in the teaching of arithmetic. Charlottesville, Va.: University of Virginia Press, 1955.

McConnell, T. R. Discovery versus authoritative identification in the learning of children. University of Iowa Studies in Education, 1934, 9, 13-62.

Nuthall, G. H. An experimental comparison of alternative instructional strategies in the teaching of concepts. Doctoral dissertation, University of Illinois, Urbana, 1966.

Rappaport, D. An investigation of the degree of understanding of meaning in arithmetic of pupils in selected elementary schools. Doctoral dissertation, Northwestern University, Evanston, 1957.

Rector, R. The relative effectiveness of four strategies for teaching mathematical concepts. Doctoral dissertation, University of Illinois, Urbana, 1966.

Retzer, K. A. The effect of teaching certain concepts of logic on verbalization of discovered mathematical generalizations. Doctoral dissertation, University of Illinois, Urbana, 1967.

Rollins, J. H. A comparison of three stratagems for teaching mathematical concepts and generalizations by guided discovery. Doctoral dissertation, University of Illinois, Urbana, 1966.

Shipp, D. E. & Deer, G. H. The use of class time in arithmetic. Arithmetic Teacher, 1960, 7, 117-121.

Shuster, A. H. & Pigge, F. L. Retention efficiency of meaningful teaching. Arithmetic Teacher, 1965, 12, 24-31.

Smith, E. P. & Henderson, K. Proof. In The growth of mathematical ideas: Grades K-12 (twenty-fourth yearbook of the National Council of Teachers of Mathematics). Washington, D.C.: National Council of Teachers of Mathematics, 1959.

Smith, B. O. Teachers for the real world. Washington, D.C.: The American Association of Colleges for Teacher Education, 1969.

Smith, B. O. et al. A study of the strategies of teaching. (Cooperative Research Project no. 1640, U. S. Office of Education.) Urbana, Ill.: Bureau of Educational Research, College of Education, University of Illinois, 1964.

Stephens, J. M. Transfer of learning. In Gross, R. F. & Birney, R. C. (Eds.), Transfer of learning. Princeton, N.J.: Van Wostrand, 1963.

Thiele, C. L. The contribution of generalization to the learning of additive facts. Teachers College Contributions to Education. New York: Teachers College, Columbia University, 1938.

Van Engen, H. The formation of concepts. In Twenty-first yearbook of the National Council of Mathematics. Washington, D.C.: National Council of Mathematics, 1953.

Wehlage, G. & Anderson, E. Social studies curriculum in perspective--A conceptual analysis. Englewood Cliffs, N.J.: Prentice-Hall, 1972.

Wills, H. Generalizations. In The teaching of secondary school mathematics (thirty-third yearbook of the National Council of Teachers of Mathematics). Washington, D.C.: National Council of Teachers of Mathematics, 1970.

Wolfe, R. Strategies of justification used in the classroom by teachers of secondary school mathematics. Doctoral dissertation, University of Illinois, Urbana, 1969.

Previous Issues in the Series:

Report
Number

1. Gliessman, David An Introduction to Protocol and Training Materials. September, 1972.

   Hudgins, Bryce B. The Portrayal of Concepts: An Issue in the Development of Protocol Materials. September, 1972.

2. Okey, James R. and Ciesla, Jerome L. Designs for the Evaluation of Teacher Training Materials. October, 1972.

3. Smith, B. Othanel and Orlosky, Donald E. The Development of Protocol Materials. January, 1973.

4. Okey, James R. and Brown, Jerry L. Competencies for Performance-Based Teacher Training. November, 1973.

   Levie, W. Howard Technical Quality and Color in Training Films. November, 1973.

5. Hudgins, Bryce B. Self-Contained Training Materials for Teacher Education: A Derivation from Research on the Learning of Complex Skills. February, 1974.

6. Kleucker, Joy Effects of Protocol and Training Materials. September, 1974.

A limited supply of some of the above reports is still available. While the supply lasts, single or multiple copies may be obtained by writing to:

National Center for the Development of
Training Materials in Teacher Education
Indiana University
2805 E. Tenth Street
Bloomington, Indiana 47401